Hierarchy of the parasupersymmetric Hamiltonians in quantum mechanics

1991 J. Phys. A: Math. Gen. 24 L529

(http://iopscience.iop.org/0305-4470/24/10/007)

J. Phys. A: Math. Gen. 24 (1991) L529–L534. Printed in the UK

LETTER TO THE EDITOR

Hierarchy of the parasupersymmetric Hamiltonians in quantum mechanics

V P Spiridonov†

Institute for Nuclear Research of the USSR Academy of Sciences, 60th October Anniversary pr. 7a, Moscow 117312, USSR

Received 18 February 1991

Abstract. The simplest parasupersymmetric model found in one-dimensional quantum mechanics is considered. It contains one bosonic and one parafermionic degree of freedom. By successive restriction of the superpotentials we show how the algebra of symmetry is simplified. Some discussion of the general parasupersymmetry algebras is presented.

In this letter we consider one-dimensional quantum mechanical systems exhibiting a symmetry which is naturally described not with the help of usual Lie algebras or their supersymmetric (SUSY) extension but by polynomial algebras. These symmetries naturally extend the notion of SUSY because they describe non-trivial mapping into each other of the degrees of freedom with different exotic statistics (i.e. neither bosonic nor fermionic). One non-trivial example of such a model is given in [1] where the symmetry between bosonic and second-order parafermionic degrees of freedom was described. Because corresponding symmetry algebra was trilinear in the conserved charges and contained as a particular case the SUSY algebra itself it was suggested to name this algebra as a parasupersymmetry (PSUSY) algebra. For a summary of some achievements on this subject see [2, 3].

Parafermi and parabose statistics [4, 5] are natural extensions of the usual Fermi and Bose ones. They describe irreducible representations of the permutation group given by the triangular Young tables. Keeping in mind elegant construction of SUSY quantum mechanics [6] one may ask whether it is possible to have analogous construction for systems with the aforementioned exotic statistics. Such a question was analysed a long time ago in the context of the parastring models [7] but the specific symmetry algebra was the usual SUSY one, because the number of parabosonic and parafermionic degrees of freedom were chosen to be equal as is required by SUSY. Below we shall present the simplest second-order PSUSY models, where there is no such equality between numbers of degrees of freedom.

The $p=2$ parafermion a, a^+ is defined by the trilinear relations

$$a^3=0 \qquad aa^+a=2a \qquad a^2a^+ + a^+a^2 = 2a. \tag{1}$$

† Present address: Laboratoire de Physique Nucleaire, Université de Montreal, CP6128, Succ. A, Montreal, Quebec, Canada H3C 3J7.

0305-4470/91/100529+06$03.50

Their matrix realization is given by

$$a^+=\sqrt{2}\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix}\qquad a=\sqrt{2}\begin{pmatrix}0&0&0\\1&0&0\\0&1&0\end{pmatrix}$$
$$[a^+,a]=2\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}=2J_3. \tag{2}$$

One has eight independent combinations of the operators a, a^+, a^2, $(a^+)^2$, a^+a, aa^+, a^+a^2, $(a^+)^2a$. Therefore the general Hamiltonian describing the interaction of a one-dimensional Bose particle with this parafermion

$$H=\tfrac{1}{2}p^2+U(x,a,a^+) \tag{3}$$

will contain in general nine independent real potential functions. Below we shall work with the $\mathbb{Z}_2$-graded systems which are realized by the removing of all odd powers of a, a^+ from (3). Then the Hamiltonian will be an even element. We can interpret the operators a, a^+ as spin variables of our particle and look at (3) as on the Hamiltonian of a $s=1$ particle in external fields with exotic spin interaction.

The question which was addressed and resolved in [1] is whether it is possible to find such potentials in (3) that we shall have the symmetry leading to the degeneracies and to the mapping of bosonic and parafermionic degrees of freedom into each other. Let us consider in more detail the corresponding model when there is only one conserved Hermitian parasupercharge Q_1 obeying the algebraic relation

$$Q_1^3=Q_1H \tag{4}$$

where H is a Hamiltonian. One can note that Q_1 is conserved as a direct result of the definition (4). Of course in this case there are no grounds to expect the degeneracy of the spectrum, but at least we may analyse eigenfunctions of the operator Q_1 instead of the eigenfunctions of a Hamiltonian. For non-zero modes the energy will be equal to the square of the eigenvalue of Q_1 and this might be of interest.

Let us describe the most general even Hamiltonian we were able to find. Parasupercharge is an odd element and has the form

$$Q_1=\tfrac{1}{2}\begin{pmatrix}0&M_1&0\\M_1^*&0&M_2\\0&M_2^*&0\end{pmatrix}=\frac{1}{4\sqrt{2}}((a^+)^2aM_1+a(a^+)^2M_2+\text{HC}) \tag{5}$$

where M_1 and M_2 are linear differential operators. Their most general form is given by

$$M_1=\alpha(p-\mathrm{i}W_1(x))\qquad M_2=\beta(p-\mathrm{i}W_2(x)) \tag{6}$$

where W_1 and W_2 are arbitrary superpotentials. The charge Q_1 contains additional free parameters α and β with respect to the charge analysed in [1] and leads to the Hamiltonian

$$\begin{aligned}4H={}&2p^2+|\alpha|^2W_1^2+|\beta|^2W_2^2\\&+\begin{pmatrix}2W_1'+|\beta|^2(W_1+W_2)'&0&0\\0&|\beta|^2W_2'-|\alpha|^2W_1'&0\\0&0&-2W_2'-|\alpha|^2(W_1+W_2)'\end{pmatrix}\\&+\frac{(W_2^2-W_1^2)'+(W_1+W_2)''}{W_1+W_2}\begin{pmatrix}|\beta|^2&0&-\alpha^*\beta^*\\0&0&0\\-\alpha\beta&0&|\alpha|^2\end{pmatrix}\end{aligned} \tag{7}$$

where we normalized $|\alpha|^2+|\beta|^2=2$ in order to have the correct coefficient in front of the kinetic term. There are no restrictions on the superpotentials in (7) (at $W_1+W_2=0$ the second row is simply absent). One can write (7) in terms of the parafermionic operators and check that it gives an exotic nonlinear spin interaction with external potentials.

If we restrict the Hamiltonian to be diagonal, i.e. put the constraint

$$(W_2^2-W_1^2)'+(W_1+W_2)''=0 \tag{8}$$

then there will appear an additional conserved Hermitian charge, Q_2, which together with Q_1 forms the algebra [1]

$$(\{Q_i, Q_j\}-2\delta_{ij}H)Q_k+\text{symmetry over }(ijk)=0. \tag{9}$$

The $N=1$ SUSY algebra is a particular solution of (9) because it corresponds to the nullification of the factored brackets. In terms of the $Q=Q_1+\mathrm{i}Q_2$ and $Q^+=Q_1-\mathrm{i}Q_2$ (9) is reduced to

$$Q^3=0 \qquad Q^2Q^++QQ^+Q+Q^+Q^2=4QH \tag{10}$$

and Hermitian conjugated relations. Substituting solution of (8) $W_2^2+W_2'=W_1^2-W_1'+c$, $c=\text{constant}$, into (7) we obtain $H=\text{diag}(h_1, h_2, h_3)$,

$$\begin{aligned}
2h_1&=p^2+W_1^2+W_1'+|\beta|^2c/2\\
2h_2&=p^2+W_1^2-W_1'+|\beta|^2c/2=p^2+W_2^2+W_2'-|\alpha|^2c/2\\
2h_3&=p^2+W_2^2-W_2'-|\alpha|^2c/2.
\end{aligned} \tag{11}$$

So, the connection with two usual SUSY Hamiltonians found in [1] remains intact and the classification of energy levels of [1] does not change (except for the shift of energy eigenvalues). An essential point consists of the fact that at $c\neq 0$ vacuum energy is not restricted and can take any negative value.

As was shown in [8] (see also [9]) condition (8) leads in fact to the existence of not two but four independent conserved Hermitian parasupercharges. They can be constructed from two objects

$$q_1=\tfrac{1}{2}\begin{pmatrix}0 & p-\mathrm{i}W_1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \qquad q_2=\tfrac{1}{2}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & p-\mathrm{i}W_2\\ 0 & 0 & 0\end{pmatrix} \tag{12}$$

e.g., the charge (5) is simply $Q_1=\alpha q_1+\beta q_2+\text{HC}$. One can easily verify the relations

$$\begin{aligned}
&q_1^2=q_2^2=0 && q_2q_1=q_2q_1^+=0 && [J_3, q_1]=q_1\\
&[J_3, q_2]=q_2 && [H, q_1]=[H, q_2]=0 && q_2q_2^+-q_1^+q_1=c.
\end{aligned} \tag{13}$$

The most important consequence of the existence of (12) is that now the Hamiltonian can be written as a bilinear combination of conserved quantities

$$H=2q_1q_1^++q_1^+q_1+q_2q_2^++2q_2^+q_2+c/4J_3+(|\beta|^2-|\alpha|^2)c/8. \tag{14}$$

In terms of the operators a, a^+ (14) holds because of the identity

$$(a^+)^2a^2+2aa^+-4=0 \tag{15}$$

which can be proven with the help of (1) and Schur's lemma. Using properties of q_1 and q_2 it is easy to check without explicit matrix representations that H commutes with them. Relations (13), (14) do not form any kind of dynamical (super)Lie algebra. A full set of polynomial (trilinear) relations found in [8] resembles the usual SUSY one and it contains in an intriguing way the constant c. It is important that a decomposition analogous to (14) does not exist for the general case (7).

Let us analyse the simplifications taking place at the condition $c=0$. It is easy to see that the Hamiltonian elements can be represented in the form

$$h_1 = M_1 M_1^+ \qquad h_2 = M_1^+ M_1 = M_2 M_2^+ \qquad h_3 = M_2^+ M_2. \tag{16}$$

Because of the middle equality the second relation of (10) is greatly simplified. Namely, it is split into the two pieces

$$Q^2 Q^+ + Q^+ Q^2 = 2QH \qquad QQ^+ Q = 2QH. \tag{17}$$

From (16) it follows that the vacuum energy is positive semidefinite and this allows us to suggest that some hidden SUSY is present. Indeed, one can check that the Hamiltonian (16) can be easily obtained by trivial cutting (deleting one row and column) of the usual one-dimensional $N=2$ supersymmetric Hamiltonian [10].

In [9, 11] it was suggested to postulate algebraic relations

$$Q^3 = 0 \qquad [Q, [Q^+, Q]] = 2QH \qquad [Q, H] = 0 \tag{18}$$

(and Hermitian conjugated) instead of the (10). Formally they are not equivalent but for the most general realization of the odd supercharge (5) at any M_1 and M_2 relations (18) exactly reduce to the particular PSUSY case given by (16) and (17). Therefore we shall not discuss (18) further.

Basic relation (1) corresponds to the specific order ($p=2$) of the parafermion a. For the general case PSUSY algebra has the following form (Q is a Hermitian charge)

$$\prod_{n=0}^{(p-1)/2} [Q^2 - (p-2n)^2 H] = 0 \qquad \text{odd } p \tag{19}$$

$$\prod_{n=0}^{(p/2)-2} [Q^2 - (p-2n)^2 H] Q = 0 \qquad \text{even } p. \tag{20}$$

These relations were obtained in a more general form in [12] (see also [3] by consideration of the diffeomorphisms on the parasuperplane (t, ϑ), where ϑ is a para-Grassmannian variable with the property $\vartheta^{p+1} = 0$. In this approach H represents the Virasoro generator L_0 and Q is one of the spin-$\frac{3}{2}$ operators.

Let us suppose that Hamiltonian H has positive spectrum and define

$$J = Q/2\sqrt{H}. \tag{21}$$

Then, the general PSUSY algebra (19), (20) can be rewritten as follows

$$(J - p/2)(J - p/2 + 1) \ldots (J + p/2 - 1)(J + p/2) = 0. \tag{22}$$

One can easily recognize in (22) the characteristic equation for the spin $p/2$ irreducible representation of the Lie algebra so(3). This means that general dynamical PSUSY algebras may be interpreted as characteristic equations of some matrices (conserved operators) with the operator valued coefficients. From this point of view the algebra of fractional SUSY [13], one of the defining relations of which has the form

$$Q^n = H \qquad n > 2 \tag{23}$$

belongs to the same set of dynamical algebras because (23) looks like the characteristic equation of the $\mathbb{Z}_n$-group.

All previously described PSUSY models referred to systems with one parafermionic degree of freedom. Let us consider the simplest parasupersymmetric Hamiltonian with a multiple number of Bose and $p=2$ parafermi creation and annihilation operators discussed in [3], [14]

$$H=\frac{1}{2}\sum_{i=1}^{N}(\{b_i^+, b_i\}+[a_i^+, a_i]) \qquad [b_i, b_j^+]=\delta_{ij} \qquad [b_i, b_j]=0 \tag{24}$$

$$a_k a_l^+ a_m + a_m a_l^+ a_k = 2\delta_{kl}a_m + 2\delta_{lm}a_k \qquad a_k^+ a_l a_m + a_m a_l a_k^+ = 2\delta_{kl}a_m \qquad a_k a_l a_m + a_m a_l a_k = 0. \tag{25}$$

Because this is a free system any object containing an equal number of creation and annihilation operators is conserved. Direct attempts to check the relation (1) or (10) with some parasupercharge showed that several conserved quantities are involved into the trilinear products and it is difficult to close the algebra. At $N>1$ a decomposition of the Hamiltonian analogous to (14) will be much more complicated because now individual a_is do not satisfy (15). Instead of (15) there will be a relation involving a product of all a_is and the form of operators similar to q_k (12) also will contain products of different a_is. For the specific case, $N=2$, a_i are 10×10 matrices and one of the PSUSY relations has the form

$$Q_1^3(Q_1^2-4H)^3-16Q_1H(Q_1^2-4H)-16Q_1=0 \qquad Q_1=\sum_{i=1}^{2} b_i^+ a_i + \mathrm{HC} \tag{26}$$

which differs from (19) and, moreover, cannot be interpreted as a characteristic equation of a 10×10 matrix. So, in the case of a multiple number of parafermionic degrees of freedom PSUSY algebras need further understanding. Analogous considerations in field theory would be highly non-local and therefore intractable. The author does not know at the moment any field theory obeying simple PSUSY relations (19) or (20).

Our last remark concerns parasupersymmetric treatment of the higher spin relativistic equations in the Duffin–Kemmer formalism [15]. The basic equation has the form

$$(\mathrm{i}\beta_\mu\partial_\mu - m)\Psi(x)=0 \tag{27}$$

where matrices β_μ coincide with usual Dirac γ-matrices for the spin-$\frac{1}{2}$ field. For the spin-1 case their commutation relations are trilinear [15]

$$\beta_\mu\beta_\nu\beta_\lambda+\beta_\lambda\beta_\nu\beta_\mu=2\delta_{\mu\nu}\beta_\lambda+2\delta_{\lambda\nu}\beta_\mu \tag{28}$$

which can be realized with the help of the Hermitian combinations of the $p=2$ parafermionic creation and annihilation operators (25). It is well known that spin-$\frac{1}{2}$ fields naturally lead to the SUSY relation between Hamiltonian H and conserved charge Q

$$H=p^2+m^2 \qquad Q=p_\mu\gamma_\mu+m\gamma_5 \qquad H=Q^2. \tag{29}$$

One can show [16] that for the spin-1 case (28) there exists an analogous charge Q which obeys with the Hamiltonian $p=2$ PSUSY relation (4) and that the relations (19), (20) naturally arise for higher spins. In fact this observation gives a very interesting realization of PSUSY for the relativistic particle and may lead to the consistent treatment of a PSUSY field theory.

Polynomial algebraic relations for the generators of the symmetries of a quantum system, being postulated as fundamental ones, may lead to the generalization of the basic theorems of quantum field theory such as Coleman–Mandula theorem and spin-statistics theorem. The example of SUSY shows that abandoning of the requirement for the algebra of symmetries to be strictly of Lie type allows one to have some non-trivial unification of the Poincaré group with internal (gauge) groups. It is natural to suggest that polynomial algebras will allow us to find more room for the unification of all forces idea. From a purely mathematical point of view PSUSY, discussed in this letter, is related to the notion of metasymmetry which recently appeared in the literature [17].

The author is indebted to V A Rubakov and L Vinet for useful discussions. I thank G P Korchemsky for drawing my attention to the similarity between PSUSY relations and the higher spin Duffin–Kemmer β-algebras.

References

[1] Rubakov V A and Spiridonov V P 1988 *Mod. Phys. Lett.* A **3** 1337
[2] Spiridonov V P *Proc. Int. Seminar 'Quarks '90', Telavi, May 1990* to appear
[3] Durand S *et al Proc. XVIII Int. Colloq. on Group Theoretical Methods in Physics, Moscow, June 1990* to appear
[4] Green H S 1953 *Phys. Rev.* **90** 270
Volkov D V 1959 *Zh. Eksp. Teor. Fiz.* **39** 1560; 1960 *Zh. Eksp. Teor. Fiz.* **38** 519
[5] Ohnuki Y and Kamefuchi S 1982 *Quantum Field Theory and Parastatistics* (University of Tokyo Press)
[6] Witten E 1981 *Nucl. Phys.* B **185** 513
[7] Ardalan F and Mansouri F 1974 *Phys. Rev.* D **9** 3341
[8] Durand S and Vinet L 1990 *Phys. Lett.* **146A** 299
[9] Beckers J and Debergh N 1990 *J. Phys. A: Math. Gen.* **23** L751
[10] Lancaster D 1984 *Nuovo Cimento* A **79** 28
Berezovoy V and Pashnev A 1989 *Teor. Mat. Fiz.* **52** 289
[11] Beckers J and Debergh N 1990 *Nucl. Phys.* B **340** 767; 1990 *J. Phys. A: Math. Gen.* **23** L1073
[12] Durand S *et al* 1989 *Phys. Lett.* **233B** 158
[13] Bernard D and Pasquier V 1989 *Saclay preprint SPhT-89/204*
[14] Floreanini R and Vinet L 1990 *Preprint UCLA/90/TEP/61*
[15] Umezawa H 1956 *Quantum Field Theory* (Amsterdam: North-Holland)
[16] *Korchemsky G P private communication*
[17] Leites D and Serganova V 1990 *Phys. Lett.* **252B** 91